

# Hair from the Isolated Horizon Perspective

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## Abstract

The recently introduced Isolated Horizons (IH) formalism has become a powerful tool for realistic black hole physics. In particular, it generalizes the zeroth and first laws of black hole mechanics in terms of quasi-local quantities and serves as a starting point for quantum entropy calculations. In this note we consider theories which admit hair, and analyze some new results that the IH provides, when considering solitons and stationary solutions. Furthermore, the IH formalism allows to state uniqueness conjectures (i.e. horizon ‘no-hair conjectures’) for the existence of solutions.

## I. INTRODUCTION

The vast majority of analytical work on black holes in general relativity centers around event horizons in globally stationary spacetimes [1]. Even when this is a natural starting point, it is not entirely satisfactory from a physical viewpoint. For instance, the collapse to form a black hole, black hole mergers, etc. are situations not described by stationary solutions. In recent years, a new framework tailored to consider situations in which the black hole is in equilibrium (nothing falls in), but which allows for the exterior region to be dynamical, has been developed. This *Isolated Horizons* (IH) formalism is now in the position of serving as starting point for several applications, from the extraction of physical quantities in numerical relativity to quantum entropy calculations [2].

The basic idea is to consider space-times with an interior boundary (to represent the horizon), satisfying quasi-local boundary conditions ensuring that the horizon remains ‘isolated’. Although the boundary conditions are motivated by geometric considerations, they lead to a well defined action principle and Hamiltonian framework. Furthermore, the boundary conditions imply that certain ‘quasi-local charges’, defined at the horizon, remain constant ‘in time’, and can thus be regarded as the analogous of the global charges defined at infinity in the asymptotically flat context. The isolated horizons Hamiltonian framework allows to define the notion of *Horizon Mass*  $M_\Delta$ , as function of the ‘horizon charges’. In the Einstein-Maxwell and Einstein-Maxwell-Dilaton systems considered originally [3], the horizon mass

satisfies a Smarr-type formula and a generalized first law in terms of quantities defined exclusively at the horizon (i.e. without any reference to infinity).

The introduction of non-linear matter fields like the Yang-Mills field brings unexpected subtleties to the formalism [4]. Firstly, the Horizon Mass can no longer be written in terms of a Smarr formula and the first law has to be reconsidered. Second, the IH formalism seems to be robust enough to allow for new results even in the static sector of the theory under consideration. The purpose of this short note is to review these results and to direct to the relevant literature where details can be found. The structure of this note is as follows. In Section II we consider the first law. In Sec. III we state the uniqueness and completeness conjectures in terms of quasi-local charges. In Sec. IV we discuss a formula relating black holes and solitons of the theory. We end with a summary in Section V

## II. THE FIRST LAW

An isolated horizon is a non-expanding null surface generated by a (null) vector field  $l^a$ . The IH boundary conditions imply that the acceleration  $\kappa$  of  $l^a$  ( $l^a \nabla_a l^b = \kappa l^b$ ) is constant on the horizon  $\Delta$ . However, the precise value it takes on each point of phase space (PS) is not determined a-priori. On the other hand, it is known that for each vector field  $t_o^a$  on spacetime, the induced vector field  $X_{t_o}$  on phase space is Hamiltonian if and only if there exists a function  $E_{t_o}$  such that  $\delta E_{t_o} = \Omega(\delta, X_{t_o})$ , for any vector field  $\delta$  on PS. This condition can be re-written as [7],

$$\delta E_{t_o} = \frac{\kappa_{t_o}}{8\pi G} \delta a_\Delta + \text{work terms} \quad (2.1)$$

Thus, the first law arises as a necessary and sufficient condition for the consistency of the Hamiltonian formulation. Thus, the allowed vector fields  $t^a$  will be those for which the first law holds. Note that there are as many ‘first laws’ as allowed vector fields  $l^a \equiv t^a$  on the horizon. However, one would like to have a *Physical First Law*, where the Hamiltonian  $E_{t_o}$  be identified with the ‘physical mass’  $M_\Delta$  of the horizon. This amounts to finding the ‘right  $\kappa$ ’. This ‘normalization problem’ can be easily overcome in the EM system [3]. In this case, one chooses the function  $\kappa = \kappa(a_\Delta, Q_\Delta)$  as the function that a *static* solution with charges  $(a_\Delta, Q_\Delta)$  has. However, for the EYM system, this procedure is not as straightforward. At present, there are two viewpoint towards this issue: i) If one wants to have a ‘global normalization’ for  $l^a$  valid on all PS, and therefore, a ‘canonical horizon mass’  $M_\Delta$ , then one has to restrict the allowed variations  $\tilde{\delta}$  to certain directions tangent to some preferred ‘leaves’ on phase space [4]. ii) One abandons the notion of a globally defined horizon mass, but then the first law is valid for arbitrary variations  $\delta$  on PS [7]. At present both viewpoints seem to be complementary.

## III. CONJECTURES

The general prescription for arriving at an explicit expression for  $M_\Delta$ , for general isolated horizons, involves the fixing of the quantity  $\kappa$  as function of the horizon parameters. For this, one requires some input from the Static solutions. The first requirement is that there be no

ambiguities. Thus we have to conjecture that (*C1*): *All static BH solutions are characterized by its horizon parameters arising from the ‘isolated horizon’ framework.* In theories where no hair is present, as is the case of the Einstein-Maxwell-Dilaton system, the number of ‘quasi-local charges’ equals the number of parameters at infinity labeling the static solutions [3]. Thus, stating a uniqueness conjecture in this theory is insensitive as to whether one is postulating it in terms of quantities at infinity (the standard viewpoint), or in terms of ‘quasi-local charges’. Our proposal is that, for general theories, one should state the postulate in terms of purely quasi-local quantities. In the EYM system, the quasi-local charges are  $a_\Delta$ , the horizon area,  $Q_\Delta$  and  $P_\Delta$ , the horizon electric and magnetic charges respectively. In this case the first conjecture *C1* reads: Given a triple of parameters  $(a_\Delta, Q_\Delta, P_\Delta)$  for which a Static solution exists, then the solution is unique. Note that this provides for a way to formulate uniqueness statements regarding black holes that was absent in the theories that admit hairy solutions.

However, this is not sufficient in order to have the Isolated Horizon framework working for the EYM system to the same extent that it works, say, for the Einstein Vacuum, EM, and EM-Dilaton systems. In order to achieve that, we would need to have a canonical normalization of  $\kappa$  for all the values of the Isolated Horizon parameters. In the previously mentioned cases this canonical choice is given by the existence of static (and spherically symmetric) Black Hole solutions for all isolated horizon values of the parameters.

For the case of the EYM system, in the regime of staticity and spherical symmetry there are, given a fixed value of  $a_\Delta$ , only a discrete set of values of  $P_\Delta$  for which there are black hole solutions. Moreover, within this regime there are no Black Hole solutions for any value of  $P_\Delta \neq 1, 0$  and  $Q_\Delta \neq 0$ . Thus if we want to have any hope that the Conjecture might be true we must formulate it outside this restrictive regime. Indeed the fact that in EYM systems there are static Black Hole solutions that are not spherically symmetric, already shows us that we must go beyond the SSS regime. In fact the solutions alluded above are axially symmetric, instead of spherically symmetric, but seem to share, with the SSS solutions, the discreetness of the allowed values of  $P_\Delta$  [6]. Thus we have to go beyond this regime as well. In fact there are strong indications (see for example the discussion in [8]) that we must go beyond the static regime, and pose the conjecture in a broad enough setting that would still allow one to single out, for a given choice of IH charges, a particular black hole solution and thus a canonical normalization of  $\kappa$ . This would be of course the class of stationary black hole solutions, where we would have to keep track also of the angular momentum, both at infinity  $J_\infty$  and at the horizon  $J_\Delta$ . The completeness conjecture would thus be: *C2: For every value of the Isolated Horizon parameters  $a_\Delta, P_\Delta, Q_\Delta, J_\Delta$  for which a space-time can be constructed, there exist also a stationary Black Hole Solution with the same value of the parameters, now characterizing the Killing Horizon.*

#### IV. HAIR AND SOLITONS

By considering the Hamiltonian formulation for Isolated black holes in the Static sector, we are lead to a formula relating HHM and ADM mass of the colored BH solutions with the ADM mass of the Solitons of the theory.

This result is arrived at by the use a general argument from symplectic geometry that

states that, within each connected component of the Static space embedded in the space of isolated horizons, the value of the Hamiltonian  $H_t$  remains constant [3]. In particular, it implies that its value is independent of the radius  $r_\Delta$  of the horizon. Thus, by considering the limit  $r_\Delta \rightarrow 0$ , we arrive at the following unexpected relation,

$$M_{\text{ADM}}^{(n)}(r_\Delta) = M_{\text{BK}}^{(n)} + \frac{1}{2} \int_0^{r_\Delta} \beta^{(n)}(\tilde{r}) \, d\tilde{r}. \quad (4.1)$$

where  $M_{\text{ADM}}^{(n)}(r_\Delta)$  is the ADM mass of the  $n$  colored black hole as function of  $r_\Delta$ ,  $M_{\text{BK}}^{(n)}$  is the ADM mass of the  $n$  soliton, and  $\beta^{(n)} = 2r_\Delta \kappa^{(n)}$ . The second term at the RHS of Eq.(4.1) is the Horizon Mass  $H_\Delta$ .

Another point provided by this type of analysis relates to the issue of the stability: It is only when  $M_{\text{ADM}} > M_\Delta$  that the solution can be unstable. One very clear example of this is given by the magnetic RN solution, which can be considered within both the Einstein Maxwell (EM) theory and the EYM theory. This solution is stable within EM but unstable within EYM [9]. We can understand this surprising fact in terms of the different values that the Horizon Mass  $M_\Delta$  takes within each theory [10]. Let us then suggest a ‘rule of thumb’ for finding potentially unstable solutions, motivated by the EYM system. In the static family of solutions, consider the limit  $r_\Delta \mapsto 0$ . We have three possibilities: i) We arrive at a regular solution with zero energy (i.e. Minkowski). This indicates that the whole family, labeled by  $r_\Delta$ , is stable; ii) There is a minimum allowed value of  $r_\Delta$  corresponding to zero surface gravity. In this case, we can not conclude anything, and; iii) In the limit one finds a regular solution with positive energy (a soliton different from the vacuum). In this case, the whole family of solutions (including the soliton) is potentially unstable. For a complete discussion of stability based on energetic considerations see [10].

## V. DISCUSSION

Let us summarize. We have studied the extension of the Isolated Horizon formalism to include the EYM system and found that it provides a powerful tool for studying some classical aspects of the theory already at the Static level. In particular, we found an unexpected relation between the ADM mass of a static spherical black hole solutions, its Horizon mass and the ADM mass of the corresponding solitonic solution, and a novel way to consider the potential instability of black holes. We have also seen that the IH formalism provides a framework in which uniqueness (‘no-hair’) conjectures can be posed, something that was absent in the standard framework based in charges at infinity.

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